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Diffeomorphisms, anomalies and the Fefferman-Graham ambiguity*

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Abstract: Using the Weyl transfomations induced by diffeomorphisms we set up a cohomological problem for the Fefferman-Graham coefficients. The cohomologically non-trivial solutions remove the ambiguity and give the non-local terms in the effective action responsible for the trace anomalies.

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Contents

1. Introduction

The AdS/CFT correspondence offers remarkable insight into non-perturbative phenomena in gauge theories [1]. Many of the proposed tests of the correspondence rely on the symmetry algebras being isomorphic.

Among the tests going beyond the mapping of the algebraic structure the correct mapping of the trace anomalies is one of the most impressive [2, 3, 4].

On the CFT side, when the theory is put in a general gravitational background, the effective action contains specific non-local terms with local Weyl variations. The trace anomalies are produced by these terms.

On the supergravity side the correspondence involves a classical calculation: one solves the equation of motion using the metric at the boundary as initial condition. The action evaluated for this classical solution gives the effective action in terms of the boundary metric. An anomaly appears in a classical calculation due to the apparently infrared logarithmically divergent terms obtained when the action is evaluated with the classical solution. In this treatment, though the anomaly is correctly reproduced, the origin of the non-local terms in the action responsible for it is not clear.

In this note we will try to answer this question. In [5] the relation between the Weyl invariance in even dimension and a certain subgroup of diffeomorphisms in odd dimension (called the "PBH transformation") was studied. It was shown that these transformations determine universally the structure of the Euler (type-A) trace anomaly. In the present note we study further the structure of the PBH transformations, in particular their implications for the structure of the Fefferman-Graham (FG in the following) expansion [6] and its physical interpretation where we will find the origin of the non-local terms.

The FG expansion deals with the solution of the equations of motion for an AdS action. The metric is expanded in powers of the radial variable and the coefficients of the expansion are determined iteratively as local expressions in terms of the boundary metric. As shown in [6] in odd dimension $d+1$, where $d=2n$, the expansion has the following features:

(i) the first $n-1$ coefficients are determined uniquely as local $2n$-dimensional diffeo-covariant expressions in terms of the boundary metric;

(ii) the expansion breaks down at order n; a logarithmic dependence on the radial variable should be introduced beginning at this order;

(iii) the n-th order coefficient is not determined completely by the equations of motion: only its trace and covariant divergence are determined iteratively as local and diffeo-invariant expressions (the "FG ambiguity").

Combining the PBH transformations and (iii) we set up a cohomological problem involving the expansion coefficients. We show that exactly the n-th order coefficient has a non-trivial cohomology and we classify it. Then features (i) and (ii) listed above are shown to be consequences of the cohomological structure uncovered. Using the solution of the cohomology problem the complete expression for the n-th order coefficient is obtained which is non local.

The gravitational action needs boundary terms. The boundary action was evaluated in [7, 8] in terms of the FG coefficients. When the non-local terms are inserted into the boundary action they provide the non-local terms in the effective action which generate the anomalies. Since the results are algebraic, being based on the behaviour of the coefficients under the PBH transformations, we expect an analogous procedure to hold also when stringy corrections are included in the gravitational lagrangian.

Due to the non-chiral nature of the Weyl transformations it is extremely convenient to use dimensional regularization. When d is non integer, the FG expansion gives well-defined local expressions for all the coefficients. The special features happening at order n mentioned above are signaled by the appearance of poles in $d-2n$. The way in which poles in dimensional regularization are translated into cohomological features in even integer dimension was studied in [9]. We summarize this procedure, tuned to the present problem, in section 2.

In section 3 we set up the cohomological problem and we discuss the consistency conditions involved.

In section 4 we classify the solutions of the cohomological problem and their relation to the n-th order coefficient in the FG expansion.

The general conclusions we have reached and possible implications and open questions are discussed in the last section. The discussion of the special features of the cohomology related to the Euler anomaly is relegated to an appendix.

2. The relation between Weyl cohomologies and dimensional regularization

We start with the well understood case of the Weyl anomalies [9].

Consider a conformal theory coupled to an external, c-number gravitational field $g_{ij}(x)$. The effective action, after integrating out the fields of the conformal theory, is $W(g)$.

In an even dimension the anomaly can be formulated as a cohomological problem [10], i.e. searching for local Weyl variations which cannot be obtained as the variation of a local action.

If we use dimensional regularization, i.e. we consider the theory in dimension d, there are two interrelated features which make the treatment of anomalies particularly simple:

1) In d dimensions there are no non-trivial cohomologies: there is always a local expression whose variation gives the anomaly in $2n$ dimensions. These local expressions have, however, a pole in $d-2n$. We distinguish between two situations:

(i) The residue of the pole vanishes exactly in $d = 2n$ due to a "topological" identity. This is the origin of the so called "type-A anomaly"; we will continue to call more generally "type-A" expressions in dimensional regularization those which have a pole and whose residue vanishes for $d = 2n$ due to some special identity.

(ii) The residue of the pole does not vanish for $d = 2n$. We will call this situation "Type B". The effective action related to this case in $d = 2n$ will have a scale.

2) In d dimensions there are no anomalies, i.e. the effective action is exactly Weyl invariant. It follows that the aforementioned local pole terms will be accompanied by finite, non-local terms such that the sum is invariant. As a consequence the variation of the pole term is equal (but opposite in sign) to the variation of the non-local term (the anomaly).

We illustrate the above general discussion with two typical examples.

We start with "type A". In d dimensions we expand the invariant effective action around $d = 2n$. The leading term is local while the order $d-2n$ term is non local. The effective action has therefore a pole term with a local residue which vanishes in $d = 2n$ and a non-local finite term. The simplest example of this kind is in $d = 2$: in

$d = 2$ the trace anomaly appears in the correlator of two energy-momentum tensors of the conformal field theory. Using Lorentz invariance, conservation and tracelessness of the energy-momentum tensor, the structure of the correlator is fixed uniquely, up to an overall constant a ($d = 2 - \epsilon$):

$$\langle T_{ij}(q) T_{kl}(-q)\rangle = a \left\{ -\frac{(q_i q_j - \delta_{ij} q^2)(q_k q_l - \delta_{kl} q^2)}{q^{2+\epsilon}} + \right. \\ \left. + \frac{\delta_{ij}\delta_{kl} q^2 - \delta_{ik}\delta_{jl} q^2 + \delta_{ik} q_j q_l + \delta_{jl} q_i q_k - \delta_{ij} q_k q_l - \delta_{kl} q_i q_j}{\epsilon} \right\}. \tag{2.1}$$

This structure in flat space leads to an analogous one when the CFT is put in a general background g_{ij}. The generating functional $W(g)$, after the CFT is integrated out, has the expansion:

$$W(g) = -\frac{1}{4}\int d^d x \sqrt{g} R \frac{1}{\Box^{1+\epsilon/2}} R + \frac{1}{d-2}\int d^d x \sqrt{g} R\,, \tag{2.2}$$

where $E_2 = R$ is the two-dimensional Euler density. As usual in dimensional regularization, the tensors are defined in d-dimensions. $W(g)$ is Weyl invariant in d dimensions to order ϵ as is easy to verify. This can be derived directly by expanding a certain Weyl invariant expression in d dimensions, around $d = 2$. The second term, with the $0/0$ structure, signals the presence in $d = 2$ of a non-trivial cohomology in the Weyl transformations.

Once we have the expansion (2.2) we can discuss how to take the limit $d \to 2$: we simply drop the second, local term which has the $0/0$ structure, and therefore the limit is the first, finite, non-local term.

On the other hand, we can still use the local term if we are interested in the Weyl variation (anomaly) of the limit: since the total Weyl variation of (2.2) is 0, the variation of the local term is equal and opposite in sign to the variation of the limit, both variations being finite. Indeed the Weyl variation of the pole term gives $\sqrt{g} R\sigma$.

Summarizing the lessons learned from this simple example:

(a) The pole term in d dimensions with a vanishing residue signals the presence of a non-trivial cohomology in integer (even) dimensions.

(b) If a certain variation of the pole term has a well-defined limit then the pole term indicates that in integer dimension we should take a non-local expression which has the same variation. The functional dependence of the non-local term can be inferred by "completing" the pole term to a Weyl invariant expression in d dimensions.

Now we discuss the "type-B" case, again in the simplest situation where it occurs, i.e. the second Weyl anomaly in $d = 4$:

Considering a CFT around $d = 4$ in a general background, the correlator of two energy-momentum tensors gives rise to a term in the effective action of the form:

$$W_d(g) = \frac{1}{d-4} \int d^d x \sqrt{g} C^{(d)}_{ijkl} \Box^{\frac{d-4}{2}} C^{(d)ijkl} \,, \tag{2.3}$$

where $C^{(d)}$ is the Weyl tensor in d dimensions (c.f. appendix A). The expression (2.3) is Weyl invariant to order $d-4$.[1]

The expression (2.3) does not have a well-defined limit for $d \to 4$. One should modify it by a "subtraction":

$$W_d^{(\mathrm{sub})}(g) = W_d(g) - \frac{\mu^{d-4}}{d-4} \int d^d x \sqrt{g} C^{(d)}_{ijkl} C^{(d)ijkl} \,. \tag{2.4}$$

It is important to remark that the "subtraction" introduces a scale μ. Now the limit can be taken, giving in $d = 4$:

$$W_4(g) = \frac{1}{2} \int d^4 x \sqrt{g} C_{ijkl} \log\left(\frac{\Box}{\mu^2}\right) C^{ijkl} \,. \tag{2.5}$$

We can rewrite W_d as:

$$W_d(g) = \frac{1}{2} \int d^d x \sqrt{g} C^{(d)}_{ijkl} \log\left(\frac{\Box}{\mu^2}\right) C^{(d)ijkl} + \frac{\mu^{d-4}}{d-4} \int d^d x \sqrt{g} C^{(d)}_{ijkl} C^{(d)ijkl} \,. \tag{2.6}$$

The expression (2.6) is analogous to (2.2), i.e. we have a local pole term and a non-local finite term, the finite term being the limit in the integer dimension. Similarly to the "type-A" situation the finite, local Weyl variation of the non-local term can be calculated by using the fact that W_d is invariant and therefore the variation of W_4 is equal and opposite in sign to the variation of the pole term. In this particular case this gives $\sqrt{g} C_{ijkl} C^{ijkl} \sigma$.

The general conclusions we want to draw from this example for the general "type-B" situation are as follows:

(a) The presence of a pole term with a non-vanishing residue in dimension d indicates that the theory in integer dimension has to be modified by defining it after a "subtraction".

(b) After the subtraction the relevant expression can be rewritten as a sum of two terms, one finite and non local and the other a local pole term with non-vanishing residue. The limit is the non-local term.

(c) The local variation can be calculated directly from the pole term.

We are now ready to use the above lessons for the study of the structure of the FG expansion.

[1] For recent proposals to extend this expression to an exact one see [11].

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3. Consistency conditions of the FG coefficients

Consider a manifold in $d+1$ dimensions with a boundary which is topologically S_d. Following Fefferman and Graham [6], one can choose a set of coordinates in which the $d+1$-dimensional metric has the form:

$$ds^2 = G_{\mu\nu}dX^\mu dX^\nu = \frac{l^2}{4}\left(\frac{d\rho}{\rho}\right)^2 + \frac{1}{\rho}g_{ij}(x,\rho)dx^i dx^j \,. \tag{3.1}$$

Here $\mu,\nu = 1,\ldots,d+1$ and $i,j = 1,\ldots,d$. The coordinates are chosen such that $\rho = 0$ corresponds to the boundary. We will assume that g_{ij} is regular at $\rho = 0$, $g_{ij}(x,\rho=0)$ being the boundary metric.

We further assume that $g_{ij}(x,\rho)$ has a power series expansion in the vicinity of $\rho = 0$, i.e. we write

$$g_{ij}(x,\rho) = \sum_{n=0}^{\infty} g_{(n)ij}(x)\rho^n \,. \tag{3.2}$$

As shown by FG [6], when the expansion (3.2) is inserted into the equations of motion following from the Einstein action with negative cosmological constant in $d+1$ dimensions, one obtains a set of recursive equations for the $g_{(n)ij}(x)$. We summarize the features of the solutions as studied by FG:

(1) In non-integer d dimensions all the $g_{(n)ij}$'s are local, diffeo-covariant expressions which are uniquely determined in terms of the boundary metric.

(2) The functions $g_{(n)ij}$ have poles at $d = 2n$ but their covariant divergences $\nabla^i g_{(n)ij}$ and traces $g^{ij}_{(0)}g_{(n)ij}$ are finite.

(3) When d is taken to $2n$, $n > 1$, the expansion breaks down since the equations become inconsistent. Starting with the ρ^n term, a $\log(\rho)$ dependence should be added but even after adding these terms the equations of motion determine only the covariant divergence and trace of $g_{(n)ij}$.

We will study the structure of the $g_{(n)ij}$, first directly in $d = 2n$. As first conditions abstracted from the study of FG we will require that $g_{(n)ij}$ has a local covariant divergence and trace which we will call $V_{(n)i}$ and $S_{(n)}$, respectively.

Next, we will use the transformation properties of $g_{(n)ij}$ under Weyl transformations. In [5] it was shown that a certain subgroup of the diffeomorphisms in $d+1$ dimensions (called "PBH transformations") induces transformations of $g_{(n)ij}$ when the boundary metric undergoes a Weyl transformation with parameter $\sigma(x)$. The transformation has the general form:

$$\delta g_{(n)ij} = 2\sigma(1-n)g_{(n)ij} + A_{(n)ij}(\sigma)\,, \tag{3.3}$$

where $A_{(n)ij}(\sigma)$ is a local, finite, covariant functional of g_{ij}[2] and σ, linear in the infinitesimal σ. The PBH transformations determine, to a large extent, $A_{(n)ij}(\sigma)$ recursively but in this section we will not use this information.

[2]When there is no danger of confusion, we will simply denote by g_{ij} the boundary metric $g_{(0)ij}$.

The structure we want to study is contained in the three local functionals $V_{(n)i}$, $S_{(n)}$, $A_{(n)ij}$. Obviously, these functionals should fulfil some consistency conditions.

We start with $A_{(n)ij}$: if we perform a second Weyl variation on (3.3) the result should not depend on the order since the Weyl group is abelian. This gives a Wess-Zumino type condition for $A_{(n)ij}$:

$$2(1-n)\sigma_1 A_{(n)ij}(\sigma_2) + \delta_{\sigma_2} A_{(n)ij}(\sigma_1) = 2(1-n)\sigma_2 A_{(n)ij}(\sigma_1) + \delta_{\sigma_1} A_{(n)ij}(\sigma_2)\,. \tag{3.4}$$

Following from the definitions of $V_{(n)i}$ and $S_{(n)}$ as covariant divergence and trace, respectively, of $g_{(n)ij}$, their Weyl variation is given in terms of $A_{(n)ij}(\sigma)$. Using (3.3) and calculating explicitly the variations in $d = 2n$ we obtain:

$$\delta S_{(n)} = -2n\sigma S_{(n)} + g^{ij} A_{(n)ij(\sigma)}\,, \tag{3.5}$$

$$\delta V_{(n)i} = -2n\sigma V_{(n)i} + \nabla^j A_{(n)ij}(\sigma) - S_{(n)}\nabla_i\sigma\,. \tag{3.6}$$

Now we can formulate our "cohomological" problem: we search for triples of *local* functionals $V_{(n)i}$, $S_{(n)}$, $A_{(n)ij}(\sigma)$ which satisfy the consistency conditions (3.4), (3.5) and (3.6); clearly, any local functional $g_{(n)ij}$, by taking its covariant divergence, trace and Weyl variation (3.3), generates such a triple. When a consistent triple does not have a local generator, the non-local $g_{(n)ij}$ producing the triple constitutes a non-trivial solution. Of course, like in all problems of this type, the non-trivial solutions are really equivalence classes under the addition of local generators. In the above treatment we will consider all the quantities to be covariant under d-dimensional diffeomorphisms.

We illustrate the above structure for the highly degenerate $n = 1$, $d = 2$ case. The only possible V_i and S are $\partial_i R$ and R, respectively.

For A_{ij} we have as possible candidates $g_{ij}R\sigma$, $\nabla_i\nabla_j\sigma$ and $g_{ij}\Box\sigma$. The first expression does not fulfil (3.4). The other two, after using (3.5) and (3.6), give the following two consistent triples:

$$\{\partial_i R, 2R, 2g_{ij}\Box\sigma\}\,, \tag{3.7}$$

$$\left\{0, \frac{1}{2}R, g_{ij}\Box\sigma - \nabla_i\nabla_j\sigma\right\}. \tag{3.8}$$

The first triple is trivial, being generated by $g_{ij}R$; the second is a non-trivial solution corresponding to:

$$\bar{g}_{(1)ij} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}} W^P(g)\,, \tag{3.9}$$

where $W^P(g)$ is the two-dimensional, non-local Polyakov action:

$$W^P(g) = -\frac{1}{4}\int d^2x\sqrt{g}R\frac{1}{\Box}R\,. \tag{3.10}$$

Going back to our physical problem, the variation $A_{(1)ij}$ is fixed uniquely by the recurrence relations following from the PBH transformations [5] to a linear combination of the ones appearing in the above two solutions. Therefore, reintroducing the normalization in terms of the AdS radius l in $d = 2$ we have the expression:

$$g_{(1)ij} = \frac{l^2}{2}[g_{ij}R - 2\bar{g}_{(1)ij}] \tag{3.11}$$

which satisfies $\delta_\sigma g_{(1)ij} = l^2 \nabla_i \nabla_j \sigma$, as it should [5]. After this simple example we will study the general problem in the next section.

4. The general analysis of the cohomology problem

As a first step we study how $A_{(n)ij}$, which are solutions of (3.4), can be constructed. There are two basic mechanisms:

(i) If a diffeo-invariant, dimensionless functional of g_{ij}, $W(g)$, has a Weyl variation for an infinitesimal Weyl parameter σ, which can be expressed as an integral over a *local* density $D(g)$

$$\delta_\sigma W(g) = \int d^{2n}x\sqrt{g}D(g)\sigma \tag{4.1}$$

then $A_{(n)ij}$, defined as

$$A_{(n)ij} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}\delta_\sigma W(g)\,, \tag{4.2}$$

fulfils (3.4). The proof is simple. Consider the commutator between the operators:

$$\left[\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}(x)}, \delta_\sigma\right] = 2(n-1)\sigma(x)\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}(x)}\,. \tag{4.3}$$

Applying (4.3) to $W(g)$ we obtain (3.3) for a $g_{(n)ij}$ defined as:

$$g_{(n)ij} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}W(g)\,. \tag{4.4}$$

Therefore, $A_{(n)}$ will fulfil (3.4) automatically and it is local by assumption. Since, obviously, a local $W(g)$ will give a local $g_{(n)ij}$ (4.4), the interesting solutions will reflect non-local $W(g)$'s, i.e. the actions which generate the Weyl anomalies. Then $A_{(n)ij}$ is expressed in (4.2) directly in terms of the anomaly $\sqrt{g}D(g)\sigma$.

Now $D(g)$ can be either E_{2n}, the Euler characteristic ("type-A" anomaly), or an expression which together with $\sqrt{g}$ is locally Weyl invariant ("type B", whose number increases with the dimension). We will distinguish the corresponding variations by indexing them with E and B, respectively.

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We postpone the discussion of a subtlety related to the "type A" until we introduce the other conditions and we proceed with the second mechanism:

(ii) If a covariant local functional of g_{ij}, $F_{(n)ij}$, transforms homogeneously under a Weyl transformation:

$$\delta_\sigma F_{(n)ij} = 2(1-n)\sigma F_{(n)ij} \tag{4.5}$$

then

$$A_{(n)ij} \equiv F_{(n)ij}\sigma \tag{4.6}$$

satisfies (3.4) as can be easily verified.

Therefore the problem is reduced to the construction of functionals which transform homogeneously. Such functionals are directly related to the $D(g)$ which give "type-B" anomalies, discussed above. Each such functional $D^B(g)$ gives two homogeneously transforming functionals:

(a) simply define:

$$F_{(n)ij} = g_{ij} D^B\,; \tag{4.7}$$

(b) for a given D^B define:

$$B_{(n)ij} \equiv \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}} W^D \tag{4.8}$$

with W^D:

$$W^D(g) \equiv \int d^{2n}x\sqrt{g} D^B(g)\,. \tag{4.9}$$

In order to show that indeed $B_{(n)ij}$ transforms homogeneously, one simply applies the commutator (4.3) to W^D, taking into account that W^D is Weyl invariant. Following from its definition, $B_{(n)ij}$ has vanishing trace and covariant divergence since W^D is invariant under Weyl transformations and under diffeomorphisms.

Now, after having a list of solutions of (3.4), we have to study which of the solutions give rise to non-trivial cohomologies.

For the solutions mentioned under (i) we discuss now a special feature related to the "Type A" — Euler one: the variation $A^E_{(n)ij}$ related to it by (4.2) can be obtained also as the inhomogeneous term in the variation of a *local* $\bar{g}^E_{ij}$. The proof based on an explicit construction of $\bar{g}^E_{ij}$ is presented in the appendix. We conclude therefore that from the point of view of (3.4) alone, $A^E_{(n)ij}$ would be trivial.

However, calculating the covariant divergence and trace corresponding to (i) we obtain the triples

$$\left\{0, -\frac{1}{2}E_{2n}, A^E_{(n)ij}\right\}, \tag{4.10}$$

$$\left\{0, -\frac{1}{2}D^B, A^B_{(n)ij}\right\}. \tag{4.11}$$

As triples, these are cohomologically non trivial. The special feature discussed in the Appendix allows us to choose for the class (4.10) a representative

$$\left(\nabla^j \bar{g}^E_{(n)ij}, g^{ij}\bar{g}^E_{(n)ij} + \frac{1}{2}E_{2n}, 0\right), \tag{4.12}$$

i.e. transforming homogeneously under Weyl transformations. Since the trace part of the triplet is an invariant made of Weyl tensors, we can always choose a representative such that also the trace part is zero; therefore all the non-trivial cohomological information resides in the first term.

The non-trivial solutions constructed under (ii) are eliminated once we include (3.5) and (3.6). Indeed, inhomogeneous pieces in the Weyl transformation proportional to D^B themselves are eliminated since the variation of the trace should obey:

$$\delta_\sigma S = -2n\sigma S + \sigma D^B\,. \tag{4.13}$$

Taking σ to be x independent (4.13) really measures the ordinary dimension of S and if S has a well-defined dimension, (4.13) cannot hold.

The $B_{(n)ij}$ constructed above have zero trace and zero covariant divergence as we remarked and therefore, by (3.5), S transforms homogeneously. Equation (3.6) becomes:

$$\delta_\sigma V_{(n)i} = -2n\sigma V_{(n)i} + B_{(n)ij}\nabla^j\sigma - S_{(n)}\nabla_i\sigma\,. \tag{4.14}$$

However, there are no local $V_{(n)i}$ with the transformation properties given by (4.14). Therefore we are left with only the non-trivial triples constructed under (i) which are in one-to-one correspondence with the Weyl anomalies. Now we are ready to discuss the rôle of the non-trivial triples in the FG expansion and their physical interpretation.

As a general method we will use the FG solutions in dimensional regularization supplemented by the analysis of their variations in integer dimensions.

We start with the two-dimensional case already analysed directly at the end of section 3. From the PBH transformation one obtains [5]:

$$\begin{aligned} g_{(1)ij} &= \frac{l^2}{d-2}\left[R_{ij} - \frac{1}{2(d-1)}Rg_{(0)ij}\right] \\ &= \frac{l^2}{d-2}\left[R_{ij} - \frac{1}{2}Rg_{(0)ij}\right] + \frac{l^2}{2(d-1)}Rg_{(0)ij}\,. \end{aligned} \tag{4.15}$$

In the first term in (4.15) we recognize $\frac{X_{(1)ij}}{d-2}$ defined in (A.3), signaling the presence of the non-trivial solution (4.10). Therefore we can directly write down the solution (3.11) for $g_{(1)ij}$.

We proceed now to discuss the four-dimensional case where the special features related to the "type-B" solution first appear. We use again the solution of the PBH equations calculated in [5]:

$$\begin{aligned} g_{(2)ij} &= c_1 l^4 C^{(d)}_{klmn} C^{(d)klmn} g_{(0)ij} + c_2 l^4 C^{(d)}_{iklm} C_j^{(d)klm} + \\ &\quad + \frac{l^4}{d-4}\Bigg\{ -\frac{1}{8(d-1)}\nabla_i\nabla_j R + \frac{1}{4(d-2)}\Box R_{ij} - \frac{1}{8(d-1)(d-2)}\Box R g_{(0)ij} - \\ &\quad - \frac{1}{2(d-2)} R^{kl} R_{ikjl} + \frac{d-4}{2(d-2)^2} R_i{}^k R_{jk} + \frac{1}{(d-1)(d-2)^2} R R_{ij} + \\ &\quad + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{(0)ij} - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{(0)ij} \Bigg\} . \end{aligned} \tag{4.16}$$

Here $C^{(d)}_{ijkl}$ is the Weyl-tensor in d dimensions which transforms homogeneously under Weyl transformations (c.f. appendix for the definition of $C^{(d)}$ and other notation used below). We can rewrite (4.16) as follows:

$$\begin{aligned} g_{(2)ij} &= \bar c_1 l^4 C^{(d)}_{klmn} C^{(d)klmn} g_{(0)ij} + \bar c_2 l^4 C^{(d)}_{iklm} C_j^{(d)klm} - \frac{l^4}{16(d-3)} \frac{B^{(d)}_{(2)ij}}{d-4} + \\ &\quad + a \frac{l^4}{16(d-3)} \left(\frac{X_{(2)ij}}{d-4} - \bar g^E_{(2)ij} \right) + \frac{l^4}{16(d-3)} \left(4(d-3) \bar R_{ik} \bar R^k_j - 4 C^{(d)}_{ikjl} \bar R^{kl} \right), \end{aligned} \tag{4.17}$$

where $B^{(d)}_{(2)ij}$ is given by (4.8), D^B being in this case the only "type-B" anomaly in $d=4$, i.e.:

$$D^B_4 \equiv C^{(d)}_{ijkl} C^{(d)ijkl} \tag{4.18}$$

and

$$\bar R_{ij} \equiv \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)} g_{ij} R \right). \tag{4.19}$$

In (4.17) the solution of the PBH recursion relation allows a general coefficient a; the solution of the equations of motion for the AdS action $S = \int d^{d+1}x\sqrt{G}(R(G) - 2\Lambda)$ requires $a = 1$ and $\bar c_1 = \bar c_2 = 0$ [3]. The non-trivial, non-local solutions in $d = 4$ are signaled by the pole terms in (4.17). The $\frac{X_{(2)ij}}{d-4}$ term indicates, in complete analogy to the $d = 2$ case discussed before, the non-local type-A term corresponding to (4.10) with $n = 2$.

The new feature is the appearance of the $\frac{B^{(d)}_{(2)ij}}{d-4}$ pole term. Its residue does not vanish. Therefore, as we discussed in section 2, the limit $d \to 4$ cannot be taken: one needs a subtraction. The physical motivation for the subtraction can be understood if we consider the Weyl variation of the pole term. We obtain:

$$\delta_\sigma \frac{B^{(d)}_{(2)ij}}{d-4} = -2\sigma \frac{B^{(d)}_{(2)ij}}{d-4} + A^B_{(2)ij} - \sigma B_{(2)ij} , \tag{4.20}$$

where $A^B_{(2)ij}$ is given by (4.2) with the choice D^B_4 for the D in (4.1). As we discussed above under (i), $A^B_{(2)ij}$ corresponds to (4.11) and has a well-defined non-local solution in $d = 4$. On the other hand, the second term $\sigma B_{(2)ij}$ was discussed in (ii) and we concluded that it doesnot correspond to a consistent triple. In the above two interrelated facts we see the origin of the inconsistency in the FG expansion: the expansion should be modified such that $\sigma B_{(2)ij}$ disappears from the variation and equivalently this should provide a prescription for the "subtraction" in dimensional regularization. The modification of the expansion proposed by FG [6] is to include logarithmic terms i.e. to replace (3.2) by:

$$g_{ij}(x,\rho) = \sum_{m=0}^{\infty} g_{(m)ij}(x)\rho^m + h_{(n)ij}(x)\rho^n \log(\rho) + \cdots \tag{4.21}$$

in $d = 2n$ dimensions, where we made explicit just the first new term. The PBH transformations

$$\rho = \rho' e^{-2\sigma(x')} \simeq \rho'(1 - 2\sigma(x'))\,, \qquad x^i = x'^i + a^i(x',\rho')\,, \tag{4.22}$$

induce now a modified transformation of $g_{(n)ij}(x)$:

$$\bar{\delta} g_{(n)ij} = \delta g_{(n)ij} - 2\sigma h_{(n)ij}\,, \tag{4.23}$$

where $\delta g_{(n)ij}$ is given in (3.3) and give for the transformation of $h_{(n)ij}$:

$$\bar{\delta} h_{(n)ij} = 2(1-n)\sigma h_{(n)ij}\,. \tag{4.24}$$

From (4.24) it follows that $h_{(n)ij}$ should be one of the expressions constructed under (ii) above which transform homogeneously. In $d = 4$, by choosing therefore $h_{(2)ij} = \frac{1}{2}B_{(2)ij}$, due to the modified (4.23) the unwanted term in the transformation $g_{(2)ij}$ is cancelled and we have now the consistent solution corresponding to (4.11).

We can easily follow through what happened in terms of a "subtraction": if we apply the operator $\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}(x)}$ to (2.6) we see that introducing the logarithmic term which eliminated the second term in the variation is equivalent to "subtracting" the second term in (2.6). The solution of the cohomological problem originates now in the first term in (2.6) which engenders (4.11). We believe that the two particular cases discussed in detail above from both points of view contain the basic features of the "type-A" and "type-B" structures. Based on that, we are ready to formulate some general rules related to the FG series:

(a) In $d = 2n$ there is a non-trivial cohomology in the Weyl transformation of the $g_{(n)ij}$ coefficient. As a consequence, in the variation of the coefficient produced recursively by the PBH transformations there will be cohomologically non-trivial contributions.

(b) These contributions are signaled in dimensional regularization by pole terms in the solution: with vanishing residue in $d = 2n$ for "type A" and non-vanishing residue for "type B".

(c) The contribution to $g_{(n)ij}$ in $d = 2n$ coming from "type A" is non local and it is obtained by applying the $\frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{ij}(x)}$ operator on the effective action which generates the Euler trace anomaly.

(d) The contribution to $g_{(n)ij}$ coming from "type B", whose number increases with the dimension, needs subtractions. The coefficient of the logarithmic term $h_{(n)ij}$ in the expansion is a linear combination of homogeneously transforming expressions which are constructed again from "type-B" anomalies. There is a one-to-one correspondence between these expressions and the subtractions.

(e) After the subtractions are made, the contributions of "type B" to $g_{(n)ij}$ are non local and can be obtained by applying $\frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{ij}(x)}$ on the pieces of the effective action producing the various "type-B" trace anomalies.

We see that following the procedure above the so-called "FG ambiguity" is removed. One obtains unique, well-defined, albeit generally non-local expressions for $g_{(n)ij}$. We remark that due to the "type-B" contributions an arbitrary scale μ will appear in the solution. Of course one can still add to the solution terms transforming homogeneously which have vanishing trace and covariant divergence.

5. Discussion and conclusions

In [7, 8] the rôle of the boundary terms in the action was analysed.[3] These authors showed that the functional derivatives of the boundary action are related to the FG coefficients. Therefore, using our expressions for the FG coefficients, one can integrate the relation and obtain explicitly the contribution of the boundary to the effective action. From the discussion in section 4 it is clear that the contributions will contain $W(g)$ used in equations (4.1) and (4.4) to generate the cohomologically non-trivial solutions. These terms are exactly the non-local terms generating the anomalies. Their coefficients are fixed partially by the PBH transformations and completely by the equations of motion. We have therefore identified the mechanism which produces the non-local anomalous terms in the effective action. We stress that the calculation described above is done entirely within the framework of classical gravity and does not require any knowledge of the energy-momentum tensor of the dual CFT. In $d + 1 = 3$ dimensions an explicit calculation along these lines can be done both for the purely bosonic and the supersymmetric cases [14]. Indeed the Liouville action and its supersymmetric generalizations appear in the effective action: as it is well known the Liouville action represents the non-local anomalous Polyakov action in the conformal gauge.

[3] The significance of boundary terms in the holographic computation of two- and three-point functions of the energy-momentum tensor was analysed in [12, 13], respectively.

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The translation of an anomaly which is an eminently quantum effect into a classical calculation is not entirely unexpected: in two dimensions bosonisation achieves this for the chiral anomaly. In the case discussed here this is happening, however, for all dimensions due to the AdS/CFT duality.

Previously the coefficients of the anomaly were determined by looking at the infrared divergent terms in the bulk action. The above described procedure gives an alternative way to calculate them, directly from the FG coefficients. As discussed in section 2, these two ways of calculating the anomaly appear also on the CFT side of the equivalence. The relation between the two calculations is based on the Weyl invariance of the total effective action in dimensional regularization. In the gravitational case probably one can use directly diffeomorphisms in $d+1$ dimensions. We defer a detailed discussion of the anomaly coefficients to a further publication.

An amusing feature is the appearance of a scale due to the type-B anomalies. The existence of an arbitrary ("ultraviolet") scale in a classical calculation is rather surprising.

As we mentioned above there is still some ambiguity left, related to non-local terms which are invariant under all the symmetries. We believe these terms cannot be obtained by (anomalous) symmetry considerations but they reflect genuine dynamical stringy corrections.

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A. Nontriviality of the triple (4.10)

We discuss now in detail the proof that the Weyl variation $A^E_{(n)ij}$ can be obtained as the inhomogeneous piece in the variation of a local tensor $\bar{g}^E_{ij}$. We start by remarking that since the Weyl variation of the Euler density E_{2n}, $\delta E_{2n} = -2n\sigma E_{2n} +$ total derivative,

$$\delta_\sigma \int d^dx\sqrt{g}E_{2n} = (d-2n)\int d^dx\sqrt{g}E_{2n}\sigma\,, \tag{A.1}$$

where the curvature tensors entering E_{2n} are in d dimensions. Using (A.1) we can write $A^E_{(n)ij}$ as

$$A^E_{(n)ij} = \frac{1}{d-2n}\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}\delta_\sigma \int d^dx\sqrt{g}E_{2n} = \frac{1}{d-2n}\delta^{(\mathrm{in})}_\sigma \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}\int d^dx\sqrt{g}E_{2n}\,, \tag{A.2}$$

where in the second step we used (4.3) and "$\delta^{(\mathrm{in})}_\sigma$" denotes the inhomogeneous part of the Weyl variation. We have dropped a term which vanishes at $d=2n$.

The expression

$$X_{(n)ij} \equiv \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{ij}} \int d^d x \sqrt{g} E_{2n} \tag{A.3}$$

vanishes in $d = 2n$ when the Euler density itself becomes a total derivative. The "topological identity" this represents is:

$$\begin{aligned} &\delta^{[i_1}_{[j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_{2n-1}]}_{j_{2n-1}]} C^{[jj_{2n-1}}_{[ii_{2n-1}} C^{j_1 j_2}_{i_1 i_2} \cdots C^{j_{2n-3} j_{2n-2}]}_{i_{2n-3} i_{2n-2}]} - \\ &\quad - \frac{1}{2n} \delta^j_i \delta^{[i_1}_{[j_1} \cdots \delta^{i_{2n-1}}_{j_{2n-1}} \delta^{i_{2n}]}_{j_{2n}]} C^{[j_1 j_2}_{[i_1 i_2} \cdots C^{j_{2n-1} j_{2n}]}_{i_{2n-1} i_{2n}]} = 0 \,, \end{aligned} \tag{A.4}$$

where C^{ij}_{kl} is the Weyl tensor in $d = 2n$. In $d = 2n$, $X_{(n)ij}$ is proportional to the left hand side of (A.4). Now we can identify a "dimensionally continued" $X^{(d)}_{(n)ij}$ by replacing the $2n$-dimensional Weyl tensors in (A.4) with the d-dimensional ones $C^{(d)ij}_{kl}$ which transform homogeneously in d dimensions. The tensor $C^{(d)ij}_{kl}$ is given explicitly by:

$$\begin{aligned} C^{(d)ij}_{kl} &= R_{kl}{}^{ij} - \frac{1}{d-2} \left(\delta^j_l R^i_k + \delta^i_k R^j_l - \delta^i_l R^j_k - \delta^j_k R^i_l \right) - \\ &\quad - \frac{1}{(d-1)(d-2)} \left(\delta^i_l \delta^j_k - \delta^j_l \delta^i_k \right) . \end{aligned} \tag{A.5}$$

Then we can expand the explicit d dependence of $X^{(d)}_{(n)ij}$ around $d = 2n$, the curvature tensors continuing to be defined in d dimensions:

$$X^{(d)}_{(n)ij} = X_{(n)ij} - (d - 2n) \bar{g}^E_{(n)ij} \,, \tag{A.6}$$

where $\bar{g}^E_{(n)ij}$ is explicitly defined as a local tensor by (A.6). Since $X^{(d)}_{(n)ij}$ transforms homogeneously, we have:

$$\frac{1}{d-2n} \delta^{(\mathrm{in})}_\sigma X_{(n)ij} = \delta^{(\mathrm{in})}_\sigma \bar{g}^E_{(n)ij} \tag{A.7}$$

and therefore

$$A^E_{(n)ij} = \delta^{(\mathrm{in})}_\sigma \bar{g}^E_{(n)ij} \,. \tag{A.8}$$

We have thus shown that the variation of the functional derivative of the anomalous part of the action related to the Euler type anomaly can be integrated to a local functional. This of course doesnot mean that the variation of the action itself can be integrated to a local action: taking the functional derivative breaks explicitly the Bose symmetry between the different insertions of the background metric, a symmetry which is a built-in requirement of the action itself.

From the above proof it is clear that in dimensional regularization the "signature" of the "type-A" non-trivial solution is the term $\frac{X_{(n)ij}}{d-2n}$. In the spirit of the general mechanism discussed in section 2, such a 0/0 expression indicates the presence of a non local, cohomologically non-trivial functional in integer dimension. Calculating the Weyl variation, trace and covariant divergence of $\frac{X_{(n)ij}}{d-2n}$, we conclude that in $d = 2n$ it corresponds to the triple (4.10).

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